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Linearizing mappings for certain nonlinear diffusion equations

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Abstract. In this paper we consider the nonlinear diffusion equations of the type $u_t = x^{1-M}[x^{N-1}\lambda(u+\mu)^{-2}u_x]_x$. It is shown that linearizing point transformations do not exist. This equation can be equivalently written as a system of two and three equations, respectively. Linearizing point transformations are sought for these two auxiliary systems and the complete list is presented. These in turn may be employed to construct contact transformations which map the nonlinear diffusion equation to a linear partial differential equation. Such linearizing point transformations exist only if $N = 2 - M$ and $N = 2 + M$.

1. Introduction

We consider the generalized radial diffusion equations of the type

$$\frac{\partial u}{\partial t} = x^{1-M}\frac{\partial}{\partial x}\left[x^{N-1}f(u)\frac{\partial u}{\partial x}\right] \tag{1.1}$$

which are of considerable interest in mathematical physics. Some cases have been used to model physical situations in fields involving diffusion processes [1, 2]. In particular, equation (1.1) when $f(u) = u^n$ has a large number of applications, for both $n > 0$ ('slow diffusion') and $n < 0$ ('fast diffusion'). In addition, when $M = N = 1$ we have the well known nonlinear diffusion equation $u_t = [f(u)u_x]_x$ which among other applications, appears in problems related to plasma physics [3] and metallurgy [4].

Transformation methods (local or non-local) have been continually applied to equation (1.1) [5–10]. Furthermore, there is a great interest in finding exact similarity solutions [11–14].

We also point out that in the literature we meet equivalent forms of equation (1.1). For example, consider the point transformation [9]

$$t' = t \qquad x' = \frac{M}{M_2}x^{M/M_2} \qquad u' = c_1u + c_2 \tag{1.2}$$

where M, $M_2 \neq 0$ and $(N_2 - 2)M = (N - 2)M_2$, connects equation (1.1) and the equation

$$\frac{\partial u'}{\partial t'} = x^{1-M_2}\frac{\partial}{\partial x'}\left[x'^{N_2-1}f(u')\frac{\partial u'}{\partial x'}\right]. \tag{1.3}$$

If we set $M_2 = N_2$ in the transformation (1.2) then $x' = (2/2 + M - N)x^{(2+M-N/2)}$ and equation (1.3) becomes $u'_{t'} = x'^{1-N_2}[x'^{N_2-1}f(u')u'_{x'}]_{x'}$, where $N_2 = (2M/2 + M - N)$. This latter equation has been considered in [7, 8]. Similarly, if we set $M_2 = 1$ then $x' = Mx^M$

and equation (1.3) becomes $u'_{t'} = [x'^n f(u')u'_{x'}]_{x'}$, where $n = (N + M - 2/M)$ which has been considered in [5].

Recently Kingston and Sophocleous [15] have presented a number of results on point transformations for certain classes of partial differential equations. We use a specific theorem to show that point transformations that linearize equation (1.1) do not exist.

In [15] it is proved that the point transformation $x' = P(x, t, u)$, $t' = Q(x, t, u)$ and $u' = R(x, t, u)$ transforms the partial differential equation (PDE)

$$u'_{t'} = H'(x', t', u', u'_1, u'_2, \ldots, u'_n) \tag{1.4}$$

to the PDE

$$u_t = H(x, t, u, u_1, u_2, \ldots, u_n) \tag{1.5}$$

where $u_n = (\partial^n u/\partial x^n)$ and $u'_n = (\partial^n u'/\partial x'^n)$, $n \geqslant 2$, H and H' are polynomials in the derivatives u_n and u'_n respectively, if and only if $P = P(x, t)$, $Q = Q(t)$ and

$$H = P_x^{-1} R_u^{-1}(P_x Q_t H' + P_t R_x + P_t R_u u_x - P_x R_t). \tag{1.6}$$

For non-degenerate point transformations (the definition is given in the next section) we require that $P_x \neq 0$, $Q_t \neq 0$ and $R_u \neq 0$.

We take H equal to the right-hand side of equation (1.1) with $f(u)$ not a constant and $H' = x'^{1-M_2}[cx'^{N_2-1}u'_{x'}]_{x'}$ which is the right-hand side of equation (1.3) with $f(u') =$ constant. We substitute these forms of H and H' into the identity (1.6) and we use the relations [15, 16]

$$u'_{x'} = (P_x)^{-1}(R_u u_x + R_x)$$
$$u'_{x'x'} = (P_x)^{-3}(P_x R_u u_{xx} + P_x R_{uu} u_x^2 + (2P_x R_{ux} - P_{xx} R_u)u_x + P_x R_{xx} - P_{xx} R_x).$$

The coefficient of u_{xx} in (1.6), which is identically equal to zero, gives

$$(P_x)^{-2}(Q_t)^{-1} R_u (x^{M-N} P_x^2 f(u) - cP^{M_2-N_2} Q_t) = 0. \tag{1.7}$$

Equation (1.7) is satisfied only if $f(u)$ is not a function of u, that is, if it is a constant. However, this is a contradiction. Therefore we have shown that point transformations that linearize equation (1.1) do not exist.

Now, if we introduce the potential v, (1.1) can be written as a system of two PDEs

$$v_x = x^{M-1} u \qquad v_t = x^{N-1} f(u) u_x. \tag{1.8}$$

Also an important related equation is the integrated form of equation (1.1)

$$v_t = x^{N-1} f(x^{1-M} v_x)[x^{1-M} v_x]_x \tag{1.9}$$

where $u = x^{1-M} v_x$. Therefore if $v = F(x, t)$ satisfies equation (1.9), then $(u, v) = (x^{1-M} F_x, F)$ solve equations (1.8) and $u = x^{1-M} F_x$ solves (1.1). In addition, if $\{u(x, t), v(x, t)\}$ satisfy (1.8), then $u(x, t)$ solves (1.1) and $v(x, t)$ solves (1.9).

In this work, we consider equations (1.1), (1.8) and (1.9) with $f(u) = \lambda(u + \mu)^{-2}$. These specific equations have a number of properties. For example, for $M = N = 1$, (1.1) admits Lie–Bäcklund transformations and there exists a transformation that maps it to the linear heat equation $u_t = u_{xx}$ [17–19]. Also transformations that linearize (1.1) exist when $N = 2 - M$ or $N = 2 + M$ [5, 10, 19]. In fact, in subsequent analysis we conclude that equation (1.1) can be linearized only if $N = 2 - M$ or $N = 2 + M$. We present a complete list of these linearizing transformations. We generalize the existing linearizing transformations and in addition we present some new results.

Bluman and Kumei [20] derived a method for determining invertible mappings from a nonlinear system of PDEs to a linear system of PDEs. The method is based on the existence of an infinite-parameter Lie group of transformations admitted by the nonlinear system. If such a group exists and certain criteria are satisfied [20], then using the infinite-dimensional symmetry a transformation may be constructed to link the nonlinear system of PDEs to a linear system of PDEs. The auxiliary system given by equations (1.8) admits an infinite-parameter Lie group of transformation, when $f(u) = \lambda(u+\mu)^{-2}$ and $N = 2-M$, which in turn leads to invertible mapping which linearizes equations (1.8) [10]. This latter mapping leads to a non-invertible mapping of (1.1) (with $f(u) = \lambda(u+\mu)^{-2}$, $N = 2-M$). In particular, the system

$$v_x = x^{M-1}u \qquad v_t = \frac{\lambda x^{N-1}}{(u+\mu)^2}u_x \tag{1.10}$$

admits the symmetry

$$\Gamma_\infty = x^{1-M}\Psi(\xi,t)\frac{\partial}{\partial x} - (u+\mu)^2\frac{\partial\Psi(\xi,t)}{\partial\xi}\frac{\partial}{\partial u} - \mu\Psi(\xi,t)\frac{\partial}{\partial v} \tag{1.11}$$

where $N = 2-M$, $\xi = v + (\mu/M)x^M$, if $M \neq 0$ or $\xi = v + \mu\log x$, if $M = 0$ and $y = \Psi(\xi,t)$ is an arbitrary solution of the linear heat equation

$$\frac{\partial y}{\partial t} - \lambda\frac{\partial^2 y}{\partial\xi^2} = 0. \tag{1.12}$$

Symmetry (1.11) leads to the invertible mapping

$$x' = v + \begin{cases} \dfrac{\mu}{M}x^M & M \neq 0 \\ \mu\log x & M = 0 \end{cases} \qquad t' = t \qquad u' = \frac{1}{u+\mu}$$

$$v' = \begin{cases} \dfrac{\mu}{M}x^M & M \neq 0 \\ \mu\log x & M = 0 \end{cases} \tag{1.13}$$

which transforms any solution $\{u'(x',t'), v'(x',t')\}$ of the linear system of PDEs

$$v'_{x'} = u' \qquad v'_{t'} = \lambda u'_{x'} \tag{1.14}$$

to a solution $\{u(x,t), v(x,t)\}$ of the nonlinear system (1.10) with $N = 2-M$ and hence to a solution $u(x,t)$ of equation (1.1) (with $f = \lambda(u+\mu)^{-2}$, $N = 2-M$).

Such mappings are used to construct contact transformations. For example, the symmetry (1.11) (with $\lambda = 1$, $\mu = 0$, $M = N = 1$) leads to the reciprocal mapping (double application gives the identity transformation, that is, it forms a cyclic group of order 2) $x' = v$, $t' = t$, $u' = u^{-1}$ and $v' = x$ which connects (1.10) and (1.14). In turn this leads to the one-to-one contact transformation [19]

$$\mathrm{d}x' = u\,\mathrm{d}x + \frac{u_x}{u^2}\,\mathrm{d}t \qquad \mathrm{d}t' = \mathrm{d}t \qquad u' = \frac{1}{u} \tag{1.15}$$

which maps the equation $u_t = (u^{-2}u_x)_x$ into the equation $u'_{t'} = u'_{x'x'}$. Choosing a fixed point (x_0, t_0), we have the following integrated form of the contact transformation (1.15)

$$x' = \int_{x_0}^{x} u\,\mathrm{d}x - \int_{t_0}^{t}\left(\frac{\partial}{\partial x}u^{-1}\right)_{x=x_0}\mathrm{d}t \qquad t' = t - t_0 \qquad u' = \frac{1}{u}.$$

Such transformations can be extended to diffusion equations of other types. See for example, [5, 6, 21, 22].

The aim of the present paper is to search for transformations analogous to (1.13) which map the system (1.10) into a linear system of the form (1.8) (we set $f(u) =$ constant). In the spirit of the work of Pallikaros and Sophocleous [9], who classified the point transformations which connect two equations of the form (1.1), we search for point transformations of the class

$$x' = P(x,t,u,v) \qquad t' = Q(x,t,u,v) \qquad u' = R(x,t,u,v) \qquad v' = S(x,t,u,v) \tag{1.16}$$

which map the system

$$v'_{x'} = x'^{M_2-1}u' \qquad v'_{t'} = cx'^{N_2-1}u'_{x'} \tag{1.17}$$

into the system (1.10).

Now with the introduction of the potential w, equations (1.8) yield the system

$$v_x = x^{M-1}u \qquad w_x = x^{1-N}v \qquad w_t = g(u) \tag{1.18}$$

where $f = (\mathrm{d}g/\mathrm{d}u)$. Also the following subsystems arise from the system (1.18)

$$w_x = x^{1-N}v \qquad w_t = g(x^{1-M}v_x) \tag{1.19}$$

$$w_{xx} + (N-1)x^{-1}w_x = x^{M-N}u \qquad w_t = g(u) \tag{1.20}$$

$$w_t = g(x^{N-M}w_{xx} + (N-1)x^{N-M-1}w_x). \tag{1.21}$$

When $f(u) = \lambda(u+\mu)^{-2}$ and $[g(u) = -\lambda(u+\mu)^{-1}]$, (1.18) which takes the form

$$v_x = x^{M-1}u \qquad w_x = x^{1-N}v \qquad w_t = -\frac{\lambda}{u+\mu} \tag{1.22}$$

admit infinite-parameter Lie groups of transformations which consequently lead to invertible mappings which connect (1.22) to a linear system [10]. As before, we search for point transformations which map (1.22) to a linear system of the form (1.18) ($g(u) =$ constant).

2. Point transformations that linearize equations (1.10)

We consider the point transformation

$$x' = P(x,t,u,v) \qquad t' = Q(x,t,u,v) \qquad u' = R(x,t,u,v) \qquad v' = S(x,t,u,v) \tag{2.1}$$

relating to x, t, $u(x,t)$, $v(x,t)$ and x', t', $u'(x',t')$, $v'(x',t')$, and assume that this is non-degenerate in the sense that the Jacobian

$$J = \frac{\partial(P,Q,R,S)}{\partial(x,t,u,v)} \neq 0 \tag{2.2}$$

and also that

$$\delta = \frac{\partial(P(x,t,u(x,t),v(x,t)),\, Q(x,t,u(x,t),v(x,t)))}{\partial(x,t)} \neq 0. \tag{2.3}$$

In (2.3) P and Q are expressed as functions of x and t whereas in (2.2) P, Q, R and S are to be regarded as functions of the independent variables x, t, u and v.

Using the transformation (2.1), we can express the derivatives of $u'(x',t')$ and $v'(x',t')$ in terms of u_x, u_t, v_x, v_t, x, t, u, v (see for example, [15, 16])

$$u'_{x'} = \frac{D_xRD_tQ - D_tRD_xQ}{\delta} \qquad u'_{t'} = \frac{-D_xRD_tP + D_tRD_xQ}{\delta} \tag{2.4}$$

$$v'_{x'} = \frac{D_xSD_tQ - D_tSD_xQ}{\delta} \qquad v'_{t'} = \frac{-D_xSD_tP + D_tSD_xQ}{\delta} \tag{2.5}$$

where D_x and D_t are the total derivatives with respect to x and t respectively.

We also consider the two systems of PDEs which are related by the point transformations (2.1)

$$v_x = x^{M_1-1}u \qquad v_t = \frac{\lambda x^{N_1-1}}{(u+\mu)^2}u_x \tag{2.6}$$

and

$$v'_{x'} = x'^{M_2-1}u' \qquad v'_{t'} = c x'^{N_2-1}u'_{x'} \tag{2.7}$$

where λ, μ and c are constants. Using (2.1), (2.4) and (2.5) and eliminating v_x and v_t from equations (2.6) (where we have taken $v_t = x^{N_1-1}f(u)u_x$ and where we will set $f(u) = \lambda(u+\mu)^{-2}$ at a later stage), equations (2.7) may be written as two identities of the form

$$\delta^{-1}E_1(x,t,u,v,u_x,u_t) = 0 \qquad \delta^{-1}E_2(x,t,u,v,u_x,u_t) = 0 \tag{2.8}$$

where x, t, u, v, u_x and u_t are regarded as independent variables and E_1, E_2 are determined polynomials in u_x and u_t.

Now, the coefficients of u_x^2, u_x, u_t in E_1 and u_x^2, u_t in E_2, which must be identically equal to zero, give $P_u = Q_x = Q_u = Q_v = S_u = 0$. Therefore, the point transformations (2.1) can be written in the simplified form

$$x' = P(x,t,v) \qquad t' = Q(t) \qquad u' = R(x,t,u,v) \qquad v' = S(x,t,v). \tag{2.9}$$

To ensure that the transformations are non-degenerate we need to have

$$Q_t R_u(P_x S_v - P_v S_x) \neq 0 \tag{2.10}$$

where we have used the forms of J and δ in (2.2) and (2.3) respectively. In addition, we must have

$$P_v^2 + R_v^2 \neq 0 \tag{2.11}$$

because otherwise the point transformations derived are equivalent to the transformations which were obtained in [9].

Finally, from $E_1 = 0$ we get

$$R = \frac{P^{1-M_2}(x^{M_1-1}uS_v + S_x)}{x^{M_1-1}uP_v + P_x} \tag{2.12}$$

and the coefficients of u_x and u_x^0 in $E_2 = 0$ give, respectively

$$x^{N_1-1}(P_v S_x - P_x S_v)f(u) + cP^{N_2-1}Q_t R_u = 0 \tag{2.13}$$

$$(x^{M_1-1}uS_v + S_x)P_t - (x^{M_1-1}uP_v + P_x)S_t + cQ_t P^{N_2-1}(x^{M_1-1}uR_v + R_x) = 0. \tag{2.14}$$

Now we substitute the expression of R given by equation (2.12) and $f(u) = \lambda(u+\mu)^{-2}$ into equations (2.13) and (2.14) and then set the coefficients of powers of u equal to zero. This gives us an overdetermined system which enables us to determine the functional forms of $Q(t)$, $P(x,t,v)$ and $S(x,t,v)$ and consequently obtain the desired point transformations. In the following analysis we omit any calculations which were performed with the assistance of the algebraic manipulation package REDUCE [23].

It turns out that point transformations of the form (2.9) exist only when in equations (2.6) we have $N_1 = 2 - M_1$ and in equations (2.7) $N_2 = 2 + M_2$ or $N_2 = 2 - M_2$, $(M_2 \neq 0)$ or $N_2 = 2 + 3M_2$, $(M_2 \neq 0)$. Therefore, (2.6) take the form

$$v_x = x^{M_1-1}u \qquad v_t = \frac{\lambda x^{1-M_1}}{(u+\mu)^2}u_x. \tag{2.15}$$

The variables ξ and η which appear in the following results are given by the relations

$$\xi = \begin{cases} v + \dfrac{\mu}{M_1} x^{M_1} & M_1 \neq 0 \\ v + \mu \log x & M_1 = 0 \end{cases} \qquad \eta = \begin{cases} v - \dfrac{\mu}{M_1} x^{M_1} & M_1 \neq 0,\ \mu \neq 0 \\ \dfrac{1}{M_1} x^{M_1} & M_1 \neq 0,\ \mu = 0 \\ v - \mu \log x & M_1 = 0,\ \mu \neq 0 \\ \log x & M_1 = 0,\ \mu = 0. \end{cases} \tag{2.16}$$

When $N_2 = 2 - M_2$, ($M_2 \neq 0$) the following three transformations connect equations (2.15) and the equations

$$v'_{x'} = x'^{M_2-1} u' \qquad v'_{t'} = c x'^{1-M_2} u'_{x'}. \tag{2.17}$$

$$t' = \frac{\lambda}{c} t \qquad x' = [M_2 \xi + c_1]^{1/M_2} \qquad v' = \begin{cases} c_2 x^{M_1} & M_1 \neq 0 \\ c_2 \log x & M_1 = 0 \end{cases} \tag{TR.1}$$

$$u' = \frac{k}{u + \mu} \qquad \text{where } k = \begin{cases} c_2 M_1, & M_1 \neq 0 \\ c_2, & M_1 = 0 \end{cases}$$

where c_1 and $c_2 (\neq 0)$ are constants. This latter point transformation generalizes the transformation (1.13). That is, if we set in (TR.1) $c = \lambda$, $c_1 = 0$, $c_2 = 1$, $M_1 = M$ and $M_2 = 1$ we obtain (1.13). In addition (TR.1) is a special case of the following point transformation.

$$t' = \frac{\lambda}{c} t \qquad x' = (M_2 \xi + c_1 t + c_2)^{1/M_2} \tag{TR.2}$$

$$v' = c_3 \eta \exp\left[-\left(\frac{c_1^2 t + 2 c_1 M_2 \xi}{4 \lambda M_2^2}\right)\right] + \theta(t, \xi)$$

$$u' = c_3 \left(-\frac{c_1}{2\lambda M_2} \eta + \frac{u - \mu}{u + \mu}\right) \exp\left[-\left(\frac{c_1^2 t + 2 c_1 M_2 \xi}{4 \lambda M_2^2}\right)\right] + \frac{\partial \theta}{\partial \xi}$$

where c_1, c_2 and $c_3 (\neq 0)$ are constants and the function $\theta(t, \xi)$ satisfies the linear PDE

$$\frac{\partial \theta}{\partial t} - \lambda \frac{\partial^2 \theta}{\partial \xi^2} - \frac{c_1}{M_2} \frac{\partial \theta}{\partial \xi} = 0. \tag{2.18}$$

In addition in (TR.2) we require $\mu \neq 0$. In the case where $\mu = 0$ (note the forms of ξ and η in equation (2.16)) we need to apply the transformation

$$\frac{u - \mu}{u + \mu} \to \frac{1}{u} \tag{2.19}$$

in the form of $u' (= R)$.

As we pointed out before, this latter point transformation generalizes (TR.1). If we set in (TR.2) $c_1 = 0$ and $\theta = -c_3 \xi$ we obtain (TR.1). Also note that, when $\lambda = c$, $\mu = c_1 = c_2 = \theta = 0$ and $M_1 = M_2 = c_3 = 1$, the point transformation (TR.2) leads to the well known contact transformation (1.15).

$$t' = -\frac{\lambda}{c} \frac{1}{t} \qquad x' = \left(\frac{M_2 \xi + c_1 t + c_2}{t}\right)^{1/M_2} \tag{TR.3}$$

$$v' = c_3\eta\sqrt{-\frac{ct}{\lambda}}\exp\left[\frac{(\xi + (c_2/M_2))^2}{4\lambda t}\right] + \theta(t,\xi)$$

$$u' = c_3\sqrt{-\frac{ct}{\lambda}}\left[\frac{(M_2\xi + c_2)\eta}{2\lambda M_2} + \frac{(u-\mu)t}{u+\mu}\right]\exp\left[\frac{(\xi + (c_2/M_2))^2}{4\lambda t}\right] + t\frac{\partial\theta}{\partial\xi}$$

where $\lambda c < 0$, c_1, c_2, $c_3(\neq 0)$ are constants, μ must not vanish and the function $\theta(t,\xi)$ satisfies the linear PDE

$$\frac{\partial\theta}{\partial t} - \lambda\frac{\partial^2\theta}{\partial\xi^2} + \left(\frac{M_2\xi + c_2}{M_2 t}\right)\frac{\partial\theta}{\partial\xi} = 0. \tag{2.20}$$

If $\mu = 0$ we need to apply the transformation (2.19) to (TR.3).

When $N_2 = 2 + M_2$ the following three transformations connect (2.15) and equations

$$v'_{x'} = x'^{M_2-1}u' \qquad v'_{t'} = cx'^{M_2+1}u'_{x'}. \tag{2.21}$$

$$t' = \frac{\lambda}{c}t \qquad x' = \exp[\xi + \lambda M_2 t] \qquad v' = \begin{cases} c_1 x^{M_1} & M_1 \neq 0 \\ c_1 \log x & M_1 = 0 \end{cases} \tag{TR.4}$$

$$u' = \frac{k\exp[-M_2(\xi + \lambda M_2 t)]}{u+\mu} \qquad \text{where } k = \begin{cases} c_1 M_1 & M_1 \neq 0 \\ c_1 & M_1 = 0 \end{cases}$$

where c_1 is a non-zero constant. If we set $\lambda = c = c_1 = M_1 = 1$ and $\mu = M_2 = 0$, then from the point transformation (TR.4) we construct the contact transformation

$$\frac{\mathrm{d}x'}{x'} = u\,\mathrm{d}x + \frac{u_x}{u^2}\,\mathrm{d}t \qquad \mathrm{d}t' = \mathrm{d}t \qquad u' = \frac{1}{u} \tag{2.22}$$

which maps the equation $u_t = (u^{-2}u_x)_x$ into the linear equation $u'_{t'} = x'(x'u'_{x'})_{x'}$. The integrated form of (2.22) is

$$x' = \exp\left\{\int_{x_0}^{x} u\,\mathrm{d}x - \int_{t_0}^{t}\left(\frac{\partial}{\partial x}u^{-1}\right)_{x=x_0}\mathrm{d}t\right\} \qquad t' = t - t_0 \qquad u' = \frac{1}{u}$$

where (x_0, t_0) is a fixed point.

As a second example, we consider the special case of (TR.4) where $\lambda = c = c_1 = 1$ and $\mu = M_1 = M_2 = 0$. Then the point transformation (TR.4) becomes

$$t' = t \qquad x' = \mathrm{e}^{v} \qquad v' = \ln x \qquad u' = \frac{1}{u}$$

which is a reciprocal transformation (cyclic group of order 2). This latter point transformation also leads to the contact transformation (2.22) which maps the nonlinear equation $u_t = x(xu^{-2}u_x)_x$ into the linear equation $u'_{t'} = x'(x'u'_{x'})_{x'}$.

$$t' = \frac{\lambda}{c}t \qquad x' = \exp[\xi + c_1 t] \tag{TR.5}$$

$$v' = c_2\eta\exp\left[-\frac{(c_1 - \lambda M_2)^2 t + 2(c_1 - \lambda M_2)\xi}{4\lambda}\right] + \theta(t,\xi)$$

$$u' = c_2\left[-\frac{(c_1 - \lambda M_2)\eta}{2\lambda} + \frac{u-\mu}{u+\mu}\right]\exp\left[-\frac{(c_1 + \lambda M_2)^2 t + 2(c_1 + \lambda M_2)\xi}{4\lambda}\right]$$
$$+ \exp[-M_2(\xi + c_1 t)]\frac{\partial\theta}{\partial\xi}$$

where c_1, $c_2(\neq 0)$ are constants, $\mu \neq 0$ and the function $\theta(t, \xi)$ satisfies the linear PDE

$$\frac{\partial\theta}{\partial t} - \lambda\frac{\partial^2\theta}{\partial\xi^2} + (\lambda M_2 - c_1)\frac{\partial\theta}{\partial\xi} = 0. \tag{2.23}$$

This latter transformation is a generalization of (TR.4). If we set $c_1 = \lambda M_2$ and $\theta = -c_2\xi$ in (TR.5), then it becomes identical to (TR.4).

$$t' = -\frac{\lambda}{c}\frac{1}{t} \qquad x' = \exp\left[\frac{\xi}{t}\right] \tag{TR.6}$$

$$v' = c_1\eta\sqrt{-\frac{ct}{\lambda}}\exp\left[\frac{(\xi+\lambda M_2)^2}{4\lambda t}\right] + \theta(t,\xi)$$

$$u' = c_1\sqrt{-\frac{ct}{\lambda}}\left[\frac{(\xi+\lambda M_2)\eta}{2\lambda} + \frac{(u-\mu)t}{u+\mu}\right]\exp\left[\frac{(\xi-\lambda M_2)^2}{4\lambda t}\right] + t\exp\left[-\frac{M_2\xi}{t}\right]\frac{\partial\theta}{\partial\xi}$$

where $\lambda c < 0$, $c_1 \neq 0$, $\mu \neq 0$ and the function $\theta(t, \xi)$ satisfies the linear PDE

$$\frac{\partial\theta}{\partial t} - \lambda\frac{\partial^2\theta}{\partial\xi^2} + \left(\frac{\xi+\lambda M_2}{t}\right)\frac{\partial\theta}{\partial\xi} = 0. \tag{2.24}$$

In the case where $\mu = 0$ we need to apply the transformation (2.19) to (TR.5) and (TR.6).

Finally, when $N_2 = 2 + 3M_2$, $(M_2 \neq 0)$ the following two transformations connect equations (2.15) and equations

$$v'_{x'} = x'^{M_2-1}u' \qquad v'_{t'} = cx'^{1+3M_2}u'_{x'}. \tag{2.25}$$

$$t' = \frac{\lambda}{c}t \qquad x' = (c_1t + c_2 - M_2\xi)^{-1/M_2} \tag{TR.7}$$

$$v' = \frac{c_3M_2\eta}{c_1t + c_2 - M_2\xi}\exp\left[\frac{2c_1M_2\xi - c_1^2t}{4\lambda M_2^2}\right] + \theta(t,\xi)$$

$$u' = c_3\left[\frac{\eta(c_1^2t + c_1c_2 - c_1M_2\xi + 2\lambda M_2^2)}{2\lambda} + \frac{M_2(u-\mu)}{u+\mu}(c_1t + c_2 - M_2\xi)\right]$$
$$\times\exp\left[\frac{2c_1M_2\xi - c_1^2t}{4\lambda M_2^2}\right] + (c_1t + c_2 - M_2\xi)^2\frac{\partial\theta}{\partial\xi}$$

where c_1, c_2, c_3 $(\neq 0)$ are constants, $\mu \neq 0$ and the function $\theta(t, \xi)$ satisfies the linear PDE

$$\frac{\partial\theta}{\partial t} - \lambda\frac{\partial^2\theta}{\partial\xi^2} + \frac{c_1}{M_2}\frac{\partial\theta}{\partial\xi} + \frac{2\lambda M_2}{c_1t + c_2 - M_2\xi}\frac{\partial\theta}{\partial\xi} = 0. \tag{2.26}$$

$$t' = -\frac{\lambda}{c}\frac{1}{t} \qquad x' = \left[\frac{c_1t + c_2 - M_2\xi}{t}\right]^{-1/M_2} \tag{TR.8}$$

$$v' = c_3\sqrt{-\frac{ct}{\lambda}}\left(\frac{M_2t\eta}{c_1t + c_2 - M_2\xi}\right)\exp\left[\frac{(\xi - (c_2/M_2))^2}{4\lambda t}\right] + \theta(t,\xi)$$

$$u' = c_3\sqrt{-\frac{ct}{\lambda}}\left[-\frac{\eta(c_2^2 + c_1c_2t - 2c_2M_2\xi - c_1M_2t\xi - 2\lambda M_2^2t + M_2^2\xi^2)}{2\lambda t}\right.$$
$$\left. + \frac{M_2(u-\mu)}{u+\mu}(c_1t + c_2 - M_2\xi)\right]\exp\left[\frac{(\xi - (c_2/M_2))^2}{4\lambda t}\right]$$
$$+ \frac{(c_1t + c_2 - M_2\xi)^2}{t}\frac{\partial\theta}{\partial\xi}$$

where $\lambda c < 0$, c_1, c_2, c_3 ($\neq 0$) are constants, $\mu \neq 0$ and the function $\theta(t, \xi)$ satisfies the linear PDE

$$\frac{\partial\theta}{\partial t} - \lambda\frac{\partial^2\theta}{\partial\xi^2} + \left(\frac{M_2\xi - c_2}{M_2 t}\right)\frac{\partial\theta}{\partial\xi} + \frac{2\lambda M_2}{c_1 t + c_2 - M_2\xi}\frac{\partial\theta}{\partial\xi} = 0. \tag{2.27}$$

In the case where $\mu = 0$ we need to apply the transformation (2.19) to (TR.7) and (TR.8).

3. Point transformations that linearize equations (1.22)

We consider the point transformation

$$\begin{aligned} &x' = P(x, t, u, v, w) \qquad t' = Q(x, t, u, v, w) \\ &u' = R(x, t, u, v, w) \qquad v' = S(x, t, u, v, w) \qquad w' = Z(x, t, u, v, w) \end{aligned} \tag{3.1}$$

relating x, t, $u(x, t)$, $v(x, t)$, $w(x, t)$ and x', t', $u'(x', t')$, $v'(x', t')$, $w'(x', t')$, and assume that this is non-degenerate in the sense that the Jacobian

$$J = \frac{\partial(P, Q, R, S, Z)}{\partial(x, t, u, v, w)} \neq 0 \tag{3.2}$$

and also that

$$\delta = \frac{\partial(P(x, t, u(x, t), v(x, t), w(x, t)), Q(x, t, u(x, t), v(x, t), w(x, t)))}{\partial(x, t)} \neq 0. \tag{3.3}$$

Using the transformations (3.1), we can express the derivatives of $u'(x', t')$, $v'(x', t')$ and $w'(x', t')$ in terms of x, t, u, v, w, u_x, u_t, v_x, v_t, w_x and w_t. The first derivatives of $u'(x', t')$ and $v'(x', t')$ are defined by the relations (2.4) and (2.5) respectively, and similarly for $w'(x', t')$ we have

$$w'_{x'} = \frac{D_x Z D_t Q - D_t Z D_x Q}{\delta} \qquad w'_{t'} = \frac{-D_x Z D_t P + D_t Z D_x Q}{\delta}. \tag{3.4}$$

We also consider the two systems of PDEs which are related by the point transformations (3.1)

$$v_x = x^{M_1-1}u \qquad w_x = x^{1-N_1}v \qquad w_t = -\frac{\lambda}{u+\mu} \tag{3.5}$$

and

$$v'_{x'} = x'^{M_2-1}u' \qquad w'_{x'} = x'^{1-N_2}v' \qquad w'_{t'} = cu' \tag{3.6}$$

where λ, μ, c are constants. Using (3.1), (2.4), (2.5) and (3.4) and eliminating v_x, w_x and w_t from (3.5), equations (3.6) may be written as three identities of the form

$$\begin{aligned} &\delta^{-1}E_1(x, t, u, v, u_x, u_t, v_t) = 0 \\ &\delta^{-1}E_2(x, t, u, v, u_x, u_t, v_t) = 0 \\ &\delta^{-1}E_3(x, t, u, v, u_x, u_t, v_t) = 0 \end{aligned} \tag{3.7}$$

where x, t, u, v, u_x, u_t and v_t are regarded as independent variables and E_1, E_2, E_3 are determined polynomials in u_x, u_t and v_t.

Now, the coefficients of $u_x v_t$ in E_1, E_2 and E_3 give $P_u = Q_u = S_u = Z_u = 0$. Also the coefficients of v_t in E_1 and E_2 lead to the result $Q_x = Q_v = Q_w = 0$. Therefore, the point transformations (3.1) can be written in the simplified form

$$\begin{aligned} &x' = P(x, t, v, w) \qquad t' = Q(t) \qquad u' = R(x, t, u, v, w) \qquad v' = S(x, t, v, w) \\ &w' = Z(x, t, v, w). \end{aligned} \tag{3.8}$$

From $E_1 = 0$ and $E_2 = 0$ we get respectively

$$R = \frac{P^{1-M_2}(x^{M_1+N_1-2}uS_v + vS_w + x^{N_1-1}S_x)}{x^{M_1+N_1-2}uP_v + vP_w + x^{N_1-1}P_x} \tag{3.9}$$

$$S = \frac{P^{N_2-1}(x^{M_1+N_1-2}uZ_v + vZ_w + x^{N_1-1}Z_x)}{x^{M_1+N_1-2}uP_v + vP_w + x^{N_1-1}P_x}. \tag{3.10}$$

Finally, the coefficients of v_t and v_t^0 in $E_3 = 0$ give respectively

$$(P_wZ_v - P_vZ_w)v + (P_xZ_v - P_vZ_x)x^{N_1-1} = 0 \tag{3.11}$$

$$\begin{aligned} x^{M_1+N_1-2}u\left[P_tZ_v - P_vZ_t + \frac{\lambda}{u+\mu}(P_vZ_w - P_wZ_v) + cP_vQ_tR\right] \\ +v[P_tZ_w - P_wZ_t + cP_wQ_tR] \\ +x^{N_1-1}\left[P_tZ_x - P_xZ_t + \frac{\lambda}{u+\mu}(P_xZ_w - P_wZ_x) + cP_xQ_tR\right] = 0. \end{aligned} \tag{3.12}$$

We substitute the form of R, which is given by equation (3.9), into equation (3.12). If we multiply the new form of (3.12) through by $[(u + \mu) \times$ denominator of $R]$ and then pick the coefficients of powers of u, we obtain three identities. These three identities and equation (3.11) enable us to derive the functional forms of $Q(t)$, $P(x, t, v, w)$ and $Z(x, t, v, w)$ and these in turn lead to the forms of $S(x, t, v, w)$ and $R(x, t, u, v, w)$ using (3.10) and (3.9) respectively. Consequently, the desired point transformations may be classified. In the following analysis we omit any further calculations.

It turns out that point transformations of the form (3.8) exist only if in equations (3.5) we have $N_1 = 2-M_1$ or $N_1 = 2+M_1$ and if in equations (3.6) $N_2 = 2-M_2$ or $N_2 = 2+M_2$ or $N_2 = 2 + M_2/3$.

The variables ξ and η which appear in the following analysis, are given by the relations

$$\xi = \begin{cases} v + \dfrac{\mu}{M_1}x^{M_1} & M_1 \neq 0 \\ v + \mu\log x & M_1 = 0 \end{cases} \tag{3.13}$$

$$\eta = \begin{cases} w - \dfrac{v}{M_1}x^{M_1} - \dfrac{\mu}{2M_1^2}x^{2M_1} & N_1 = 2 - M_1,\ M_1 \neq 0 \\ w + \dfrac{v}{M_1}x^{-M_1} + \dfrac{\mu}{M_1}\log x & N_1 = 2 + M_1,\ M_1 \neq 0 \\ w - v\log x - \dfrac{\mu}{2}(\log x)^2 & N_1 = 2,\ M_1 = 0. \end{cases} \tag{3.14}$$

We set $N_1 = 2 - M_1$, $(M_1 \neq 0)$ and $N_2 = 2 - M_2$, $(M_2 \neq 0)$ in equations (3.5) and (3.6) respectively, to obtain

$$v_x = x^{M_1-1}u \qquad w_x = x^{M_1-1}v \qquad w_t = -\frac{\lambda}{u+\mu} \tag{3.15}$$

and

$$v'_{x'} = x'^{M_2-1}u' \qquad w'_{x'} = x'^{M_2-1}v' \qquad w'_{t'} = cu'. \tag{3.16}$$

The following two point transformations connect equations (3.15) and (3.16).

$$t' = \frac{\lambda}{c}t \qquad x' = (M_2\xi + c_1t + c_2)^{1/M_2} \tag{TR.9}$$

$$w' = c_3\eta \exp\left[-\frac{c_1^2 t + 2c_1 M_2\xi}{4\lambda M_2^2}\right] + \theta(t,\xi)$$

$$v' = -c_3\left(\frac{c_1\eta + 2\lambda(M_2/M_1)x^{M_1}}{2\lambda M_2}\right)\exp\left[-\frac{c_1^2 t + 2c_1 M_2\xi}{4\lambda M_2^2}\right] + \theta_\xi$$

$$u' = c_3\left(\frac{c_1^2\eta + 4c_1\lambda(M_2/M_1)x^{M_1}}{4\lambda^2 M_2^2} - \frac{1}{u+\mu}\right)\exp\left[-\frac{c_1^2 t + 2c_1 M_2\xi}{4\lambda M_2^2}\right] + \theta_{\xi\xi}$$

where c_1, c_2, c_3 ($\neq 0$) are constants and the function $\theta(t,\xi)$ satisfies the linear PDE (2.18).

$$t' = -\frac{\lambda}{c}\frac{1}{t} \qquad x' = \left(\frac{M_2\xi + c_1 t + c_2}{t}\right)^{1/M_2} \tag{TR.10}$$

$$w' = c_3\eta\sqrt{-\frac{ct}{\lambda}}\exp\left[\frac{(\xi + (c_2/M_2))^2}{4\lambda t}\right] + \theta(t,\xi)$$

$$v' = c_3\sqrt{-\frac{ct}{\lambda}}\left(\frac{c_2\eta + M_2\eta\xi - 2\lambda(M_2/M_1)tx^{M_1}}{2\lambda M_2}\right)\exp\left[\frac{(\xi + (c_2/M_2))^2}{4\lambda t}\right] + t\theta_\xi$$

$$u' = c_3\sqrt{-\frac{ct}{\lambda}}\left[\frac{(c_2\eta + M_2\eta\xi - 4\lambda(M_2/M_1)tx^{M_1})(M_2\xi + c_2) + 2\lambda M_2^2 t\eta}{4\lambda^2 M_2^2} - \frac{t^2}{u+\mu}\right]$$
$$\times \exp\left[\frac{(\xi + (c_2/M_2))^2}{4\lambda t}\right] + t^2\theta_{\xi\xi}$$

where $\lambda c < 0$, c_1, c_2, c_3 ($\neq 0$) are constants and the function $\theta(t,\xi)$ satisfies the linear PDE (2.20).

The following two point transformations connect equations (3.15) and equations

$$v'_{x'} = x'^{M_2-1}u' \qquad w'_{x'} = x'^{1-M_2}v' \qquad w'_{t'} = cu' \tag{3.17}$$

where we have set $N_2 = 2 + M_2$ in equations (3.6). Note that in this case the parameter M_2 can be zero.

$$t' = \frac{\lambda}{c}t \qquad x' = \exp[\xi + c_1 t] \tag{TR.11}$$

$$w' = c_2\eta\exp\left[-\frac{(c_1 + \lambda M_2)^2 t + 2(c_1 + \lambda M_2)\xi}{4\lambda}\right] + \theta(t,\xi)$$

$$v' = \left\{-c_2\frac{2(\lambda/M_1)x^{M_1} + c_1\eta + \lambda M_2\eta}{2\lambda}\right.$$
$$\left.\times\exp\left[-\frac{(c_1 + \lambda M_2)^2 t + 2(c_1 + \lambda M_2)\xi}{4\lambda}\right] + \theta_\xi\right\}\exp[c_1 M_2 t + M_2\xi]$$

$$u' = c_2\left(\frac{4c_1(\lambda/M_1)x^{M_1} + c_1^2\eta - \lambda^2 M_2^2\eta}{4\lambda^2} - \frac{1}{u+\mu}\right)$$
$$\times\exp\left[-\frac{(c_1 + \lambda M_2)^2 t + 2(c_1 + \lambda M_2)\xi}{4\lambda}\right] + M_2\theta_\xi + \theta_{\xi\xi}$$

where c_1, c_2 ($\neq 0$) are constants and the function $\theta(t,\xi)$ satisfies the linear PDE

$$\frac{\partial\theta}{\partial t} - \lambda\frac{\partial^2\theta}{\partial\xi^2} - (c_1 + \lambda M_2)\frac{\partial\theta}{\partial\xi} = 0. \tag{3.18}$$

$$t' = -\frac{\lambda}{c}\frac{1}{t} \qquad x' = \exp\left[\frac{\xi}{t}\right] \tag{TR.12}$$

$$w' = c_1\eta\sqrt{-\frac{ct}{\lambda}}\exp\left[\frac{(\xi-\lambda M_2)^2}{4\lambda t}\right] + \theta(t,\xi)$$

$$v' = \left\{c_1\sqrt{-\frac{ct}{\lambda}}\left[\frac{\eta\xi - \lambda M_2\eta - 2(\lambda/M_1)tx^{M_1}}{2\lambda}\right]\exp\left[\frac{(\xi-\lambda M_2)^2}{4\lambda t}\right] + t\theta_\xi\right\}\exp\left[\frac{M_2\xi}{t}\right]$$

$$u' = c_1\sqrt{-\frac{ct}{\lambda}}\left[\frac{\eta\xi^2 - 4(\lambda/M_1)tx^{M_1}\xi + 2\lambda t\eta - \lambda^2 M_2^2\eta}{4\lambda^2} - \frac{t^2}{u+\mu}\right]$$
$$\times\exp\left[\frac{(\xi-\lambda M_2)^2}{4\lambda t}\right] + M_2 t\theta_\xi + t^2\theta_{\xi\xi}$$

where $\lambda c < 0$, $c_1 \neq 0$ and the function $\theta(t,\xi)$ satisfies the linear PDE

$$\frac{\partial\theta}{\partial t} - \lambda\frac{\partial^2\theta}{\partial\xi^2} + \left(\frac{\xi-\lambda M_2}{t}\right)\frac{\partial\theta}{\partial\xi} = 0. \tag{3.19}$$

Finally, we set $N_2 = 2 + (M_2/3)$, $(M_2 \neq 0)$ in (3.6) to obtain

$$v'_{x'} = x'^{M_2-1}u' \qquad w'_{x'} = x'^{1-M_2/3}v' \qquad w'_{t'} = cu'. \tag{3.20}$$

Equations (3.15) and (3.20) are connected by the point transformations (TR.13) and (TR.14) which are read as follows

$$t' = \frac{\lambda}{c}t \qquad x' = \left(\frac{c_1t + c_2 + M_2\xi}{3}\right)^{3/M_2} \tag{TR.13}$$

$$w' = c_3\frac{M_2\eta}{c_1t + c_2 + M_2\xi}\exp\left[-\frac{c_1^2t + 2c_1M_2\xi}{4\lambda M_2^2}\right] + \theta(t,\xi)$$

$$v' = -c_3\left[\frac{(c_1t + c_2 + M_2\xi)(2\lambda(M_2/M_1)x^{M_1} + c_1\eta) + 2\lambda M_2^2\eta}{18\lambda}\right]$$
$$\times\exp\left[-\frac{c_1^2t + 2c_1M_2\xi}{4\lambda M_2^2}\right] + \frac{1}{9}(c_1t + c_2 + M_2\xi)^2\theta_\xi$$

$$u' = \frac{c_3}{c_1t + c_2 + M_2\xi}\left[\frac{4c_1\lambda(M_2/M_1)x^{M_1} + c_1^2\eta}{4\lambda^2M_2} - \frac{M_2}{u+\mu}\right]$$
$$\times\exp\left[-\frac{c_1^2t + 2c_1M_2\xi}{4\lambda M_2^2}\right] + \frac{2M_2}{c_1t + c_2 + M_2\xi}\theta_\xi + \theta_{\xi\xi}$$

where c_1, c_2, c_3 $(\neq 0)$ are constants the function $\theta(t,\xi)$ satisfies the linear PDE

$$\frac{\partial\theta}{\partial t} - \lambda\frac{\partial^2\theta}{\partial\xi^2} - \frac{c_1}{M_2}\frac{\partial\theta}{\partial\xi} - \frac{2\lambda M_2}{c_1t + c_2 + M_2\xi}\frac{\partial\theta}{\partial\xi} = 0. \tag{3.21}$$

$$t' = -\frac{\lambda}{c}\frac{1}{t} \qquad x' = \left[\frac{M_2\xi + c_1t + c_2}{3t}\right]^{3/M_2} \tag{TR.14}$$

$$w' = c_3\sqrt{-\frac{ct}{\lambda}}\frac{M_2t\eta}{M_2\xi + c_1t + c_2}\exp\left[\frac{(M_2\xi + c_2)^2}{4\lambda M_2^2 t}\right] + \theta(t,\xi)$$

$$v' = c_3\sqrt{-\frac{ct}{\lambda}}\left[\frac{(c_2\eta + M_2\eta\xi - 2\lambda(M_2/M_1)tx^{M_1})(M_2\xi + c_1t + c_2) - 2\lambda M_2^2t\eta}{18\lambda t}\right]$$
$$\times\exp\left[\frac{(M_2\xi + c_2)^2}{4\lambda M_2^2 t}\right] + \frac{(M_2\xi + c_1t + c_2)^2}{9t}\theta_\xi$$

$$u' = c_3\sqrt{-\frac{ct}{\lambda}}\frac{t}{M_2\xi + c_1 t + c_2}\left[\frac{(M_2\xi + c_2)(c_2\eta + M_2\eta\xi - 4\lambda(M_2/M_1)tx^{M_1}) + 2\lambda M_2^2 t\eta}{4\lambda^2 M_2}\right.$$
$$\left. - \frac{M_2 t^2}{u+\mu}\right]\exp\left[\frac{(M_2\xi + c_2)^2}{4\lambda M_2^2 t}\right] + \frac{2M_2 t^2}{M_2\xi + c_1 t + c_2}\theta_\xi + t^2\theta_{\xi\xi}$$

where $\lambda c < 0$, c_1, c_2, c_3 $(\neq 0)$ are constants and the function $\theta(t, \xi)$ satisfies the linear PDE

$$\frac{\partial\theta}{\partial t} - \lambda\frac{\partial^2\theta}{\partial\xi^2} + \frac{M_2\xi + c_2}{M_2 t}\frac{\partial\theta}{\partial\xi} - \frac{2\lambda M_2}{M_2\xi + c_1 t + c_2}\frac{\partial\theta}{\partial\xi} = 0. \tag{3.22}$$

As we stated earlier point transformations of the form (3.8) also exist when we have $N_1 = 2 + M_1$, $(M_1 \neq 0)$ in (3.5)

$$v_x = x^{M_1 - 1}u \qquad w_x = x^{-(M_1+1)}v \qquad w_t = -\frac{\lambda}{u+\mu}. \tag{3.23}$$

If $N_1 = 2 + M_1$ $(M_1 \neq 0)$ then making the following changes

$$\xi \to M_1\eta \qquad \eta \to \xi \qquad x^{M_1} \to -M_1 x^{M_1} \qquad \frac{1}{u+\mu} \to -\frac{M_1 x^{M_1}}{u+\mu} \tag{3.24}$$

where the variables ξ and η are given by the relations (3.13) and (3.14) respectively, on transformations (TR.9)–(TR.14) we have the following results.

(a) The point transformations (TR.9) and (TR.10) connect equations (3.23) and (3.16);

(b) the point transformations (TR.11) and (TR.12) connect equations (3.23) and (3.17);

(c) the point transformations (TR.13) and (TR.14) connect equations (3.23) and (3.20).

If we apply the transformation (3.24) on (TR.9) and if we set $M_1 = M_2 = \lambda = c = c_3 = 1$, $\mu = c_1 = c_2 = 0$ and $\theta =$ constant, then the resulting point transformation

$$t' = t \qquad x' = w + \frac{v}{x} \qquad w' = v + \theta \qquad v' = x \qquad u' = \frac{x}{u} \tag{3.25}$$

leads to the contact transformation

$$\mathrm{d}x' = \frac{u}{x}\,\mathrm{d}x + \left(\frac{xu_x}{u^2} - \frac{1}{u}\right)\mathrm{d}t \qquad \mathrm{d}t' = \mathrm{d}t \qquad u' = \frac{x}{u} \tag{3.26}$$

which maps the nonlinear PDE $u_t = (x^2u^{-2}u_x)_x$ into the linear PDE $u'_{t'} = u'_{x'x'}$.

As a second example, we apply the transformation (3.24) on the point transformation (TR.11) and then we set $M_1 = \lambda = c = c_2 = 1$ and $M_2 = \mu = c_1 = \theta = 0$ to give

$$t' = t \qquad x' = \mathrm{e}^{w+(v/x)} \qquad w' = v \qquad v' = x \qquad u' = \frac{x}{u}. \tag{3.27}$$

This latter point transformation leads to the contact transformation

$$\frac{\mathrm{d}x'}{x'} = \frac{u}{x}\,\mathrm{d}x + \left(\frac{x^2u_x}{u^2} - \frac{1}{u}\right)\mathrm{d}t \qquad \mathrm{d}t' = \mathrm{d}t \qquad u' = \frac{x}{u} \tag{3.28}$$

which maps the nonlinear PDE $u_t = (x^2u^{-2}u_x)_x$ into the linear PDE $u'_{t'} = x'(x'u'_{x'})_{x'}$.

Finally if $N_1 = 2$ and $M_1 = 0$ then equations (3.5) take the form

$$v_x = \frac{u}{x} \qquad w_x = \frac{v}{x} \qquad w_t = -\frac{\lambda}{u+\mu} \tag{3.29}$$

and making the change

$$\frac{1}{M_1}x^{M_1} \to \log x \tag{3.30}$$

in transformations (TR.9)–(TR.14), where the variables ξ and η are given by the relations (3.13) and (3.14) respectively, we have the following results.

(a) The point transformations (TR.9) and (TR.10) connect equations (3.29) and (3.16);

(b) the point transformations (TR.11) and (TR.12) connect equations (3.29) and (3.17);

(c) the point transformations (TR.13) and (TR.14) connect equations (3.29) and (3.20).

If we apply (3.30) on (TR.11) and set $\lambda = c = 1$, $c_2 = -1$ and $M_2 = \mu = c_1 = \theta = 0$, then we obtain the reciprocal transformation (cyclic group of order 2)

$$t' = t \qquad x' = \mathrm{e}^{v} \qquad w' = -w + v \log x \qquad v' = \log x \qquad u' = \frac{1}{u}. \tag{3.31}$$

4. Remarks

Probably the most useful and powerful point transformations of PDEs are those which form continuous (Lie) group transformations, each member of which leaves an equation invariant. Such transformations are not appropriate for directly linking a PDE with an equation of a different form. This is useful, for example, when converting equations to a canonical form on which an established theory can be used. This was the aim of the present paper, to connect a class of nonlinear PDEs to a class of linear PDEs. Furthermore, the point transformation analysis, in addition to transformations which link different equations and transformations that lead to symmetries, might produce discrete symmetries which have been overlooked in the classical method for determining symmetries of a specific class of PDEs. For example, the discrete symmetry [16] $x' = x/t$, $t' = 1/t$, $u' = -(ut - x)$ leaves the Burger-type equation $u_t + uu_x + (f(t) - f(1/t))u_{xx} = 0$ invariant, an additional symmetry to the Lie point ones obtained from the classical approach [24].

These reasons show that there is merit in studying point transformations directly in finite forms with the ultimate dual aims of finding the complete set of point transformation symmetries of PDEs and also discovering new links between different equations.

The results which are presented in this paper may be seen as a part of the problem of classifying the complete list of point transformations of the class (2.1) that connect the two systems of PDEs

$$v_x = x^{M_1-1}u \qquad v_t = x^{N_1-1}f(u)u_x \tag{4.1}$$

and

$$v'_{x'} = x'^{M_2-1}u' \qquad v'_{t'} = x'^{N_2-1}g(u')u'_{x'} \tag{4.2}$$

and of the class (3.1) that connect the two systems of PDEs

$$v_x = x^{M_1-1}u \qquad w_x = x^{1-N_1}v \qquad w_t = F(u) \tag{4.3}$$

and

$$v'_{x'} = x'^{M_2-1}u' \qquad w'_{x'} = x'^{1-N_2}v' \qquad w'_{t'} = G(u') \tag{4.4}$$

where $f = \mathrm{d}F/\mathrm{d}u$ and $g = \mathrm{d}G/\mathrm{d}u'$. Symmetries for the systems (4.1) and (4.3) have already been considered in [10]. Also the complete list of point transformations of equation (1.1) is presented in [9]. Further study, along the lines of this paper, in classifying the point transformations of the systems (4.1) and (4.3) may therefore be useful.

As stated earlier, the main feature for point transformations relating different PDEs is that using known solutions for one you can generate new solutions for the other. In our case, since, in principle, it is easier to solve linear PDEs, the point transformations presented here may be employed to generate new solutions for the nonlinear systems of PDEs (1.10) and

(1.22) using known solutions for the corresponding linear system of PDEs. Furthermore, the contact transformations obtained from these point transformations may also be employed to generate new solutions for the nonlinear PDE (1.1) ($f(u) = (\lambda/(u+\mu)^2)$) using solutions of the linear PDE (1.3) ($f(u') =$ constant). Such examples of using linearizing contact transformations to generate new solutions can be found in [5, 18, 19, 21, 22].

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